Notation: $\mathbb{R}^n$ is n-dimensional space.
i.e. $\mathbb{R}$ is $(-\infty, \infty)$
$\mathbb{R}^2$ is the plane
etc.
We will usually focus on $\mathbb{R}$ or $\mathbb{R}^3$.

Lecture 1: Motivation + Introduction

- Recall that an ordinary differential equation is an equation involving the derivative of a function. We can solve some ODE's

  e.g. $\begin{cases} y'' = y \\ y(0) = 0 \end{cases}$ has solution $y(x) = \sin(x)$

  $\rightarrow$ Note: y is a function $y : \mathbb{R} \to \mathbb{R}$

- A Partial differential equation is the multivariable analogue

  Such as $\cancel{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 0}$ $\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \\ u(0,0) = 0 \end{cases}$ for $u : \mathbb{R}^2 \to \mathbb{R}$.

- One of the earliest PDE's was the wave equation in space-time. Let $u(t,x)$ denote the position of a string (vertical displacement) at time $t > 0$ and location x. Then, u satisfies $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$. If we hold the string at both ends, we get initial conditions $u(t, 0) = u(t, \ell) = 0$, where ℓ is the length of the string.

  We will later solve this equation to find a family of solutions
  $$u(t,x) = f_1(x+t) + f_2(x-t) \quad \text{for } f_1 \text{ \& } f_2$$
  twice-differentiable

- We will study other examples such as
  - the heat equation $\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$ } describes heat flow
  - linear transport equation $\frac{\partial u}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = 0$ (b_i are scalars)
  - Laplace's Eqn $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$

$\rightsquigarrow$ Our goal is to model physical phenomena by a PDE and solve to understand physical behaviors, like heat exchange

General Notation

0.) For $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots x_n)$ as a vector.

1.) u will usually be a function we are focused on for the PDE

2.) The domain of u is some subset U of $\mathbb{R}^n$ (space) or
$\mathbb{R}_t \times \mathbb{R}^n$ (spacetime)
↳ this $\mathbb{R}_t$ denotes our time variable t

Then, we have $u : U \to \mathbb{R}$.

3.) Partial derivatives may be denoted $\frac{\partial u}{\partial x_i} = \partial_{x_i} u = \partial_i u = u_{x_i}$

4.) A general PDE may be written

$$F\left(x, u(x), \frac{\partial u}{\partial x_j}, \dots, \frac{\partial^m u}{\partial x_{j_1} \cdots \partial x_{j_m}}\right) = 0 \qquad (A)$$

- If the highest derivative appearing is of order m, the PDE is of order m

ex.) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ is of order 2

5.) Recall that $u(x)$ is continuous at a if

$$\lim_{x \to a} u(x) = u(a)$$

if u has continuous partial derivatives of order m, we write $u \in C^m(U; \mathbb{R})$ or just $u \in C^m(U)$.

6.) A classical solution on U to (A) is a function $u \in C^m(U)$ solving (A) (including given initial conditions)

lassification

- PDEs are difficult and diverse. we group them to understand them better.

1.) Linear PDE: A linear PDE is linear in u, so it may be written $Lu = f$ for L a differential operator. For our purposes, we will focus on order at-most 2.

Thus,
$$L = \underbrace{-\sum_{i,j=1}^{n}}\, a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c$$
for $a_{ij}, b_i, c \in \mathbb{R}$ ↳ why the negative? It helps in solving when using integration-by-parts (such as the energy method).

- linearity allows us to add and subtract solutions to get new solutions. This lets us decompose problems ~~using~~ into simpler components
- They're also just easier to solve

e.g.) All our examples above!

2.) Evolution Equations involve development over time. Two important classes are
   A.) Hyperbolic - "like the wave equation"
   B.) Parabolic - "like the heat equation"

3.) Elliptic - "like the Laplace Equation"

Well-Posedness

- A well-posed PDE problem is one where, given a sufficiently "nice" set of input data (initial or boundary conditions) a solution exists, is uniquely determined by the data, and continuously determined by the data.
  ↳ we will discuss ~~this~~ what this means later.

- well-posed problems are stable and solvable when they arise in various situations.